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On hyperbolicity in the Cartesian Sum of Graphs

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) = \text{ínf}\{\delta \geqslant 0 : X \text{ is } \delta\text{-hyperbolic}\}$. Some previous works characterize the hyperbolic product graphs (for the Cartesian, strong, join, corona and lexicographic products) in terms of properties of the factor graphs. In this paper we characterize the hyperbolic product graphs for the Cartesian sum $G_1 \oplus G_2$: $G_1 \oplus G_2$ is always hyperbolic, unless either G_1 or G_2 is the trivial graph (the graph with a single vertex); if G_1 or G_2 is the trivial graph, then $G_1 \oplus G_2$ is hyperbolic if and only if G_2 or G_1 is hyperbolic, respectively. Besides, if $t \notin \{5/4, 3/2\}$ we characterize the Cartesian sums with $\delta(G_1 \oplus G_2) = t$ in a very simple way; also, we characterize the Cartesian sums with $\delta(G_1 \oplus G_2) = 5/4$ and with $\delta(G_1 \oplus G_2) = 3/2$. We obtain the sharp inequalities $1 \leqslant \delta(G_1 \oplus G_2) \leqslant 3/2$ for every non-trivial graphs G_1, G_2. Furthermore, we obtain simple formulae for the hyperbolicity constant of the Cartesian sum of many graphs. Finally, we prove the inequalities $3/2 \leqslant \delta(\overline{G_1 \oplus G_2}) \leqslant 2$ for the complement graph of $G_1 \oplus G_2$ for every G_1, G_2 with $\text{mín}\{\text{diam}V(G_1), \text{diam}V(G_2)\} \geqslant 3$.